A Converse Landing Theorem in Parameter Spaces

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Abstract

In this article, under very general conditions for one parameter holomorphic families of holomorphic maps, for a given parabolic parameter, we give a transparent proof for landing of (a) parameter ray(s) at the parabolic parameter in question. Moreover, we partially answer the question of existence of rays in parameter spaces.

1 Introduction

We consider the dynamical system obtained by the iterates of holomorphic maps. We are interested in understanding the long term behavior of orbits of points -sequences generated by the iterates- in terms of their initial conditions. If the orbit of a point z_0 under a holomorphic map f is finite, we say that z_0 is pre-periodic. If there exists $q \in \mathbb{N}$ satisfying $f^q(z_0) = z_0$, then z_0 is periodic; in particular, z_0 is a fixed point if $q = 1$. Periodic points are classified in terms of their multipliers $\mu(z_0) := (f^k)'(z_0)$: If $|\mu(z_0)| < 1$, z_0 is an attracting, if $|\mu(z_0)| > 1$, z_0 is a repelling, and if $|\mu(z_0)| = 1$, z_0 is an indifferent periodic point. We are interested in the case $\mu(z_0) = e^{2\pi i k}$ and k is a rational number, i.e., z_0 is a parabolic periodic point. In this case there are some neighborhoods where points are attracted to z_0, and some where the points are repelled under the qth iterate. These are called *attracting and repelling petals*, respectively. The map f^q is conjugate to translation $T_1 : w \mapsto w + 1$, by a pair of univalent maps in attracting and repelling petals, which are called *the Fatou coordinates*.

The dynamical plane for a homolorphic map f is partitioned into two totally invariant sets with respect to the behavior of the points; the set of points with stable behavior, i.e., the domain of normality of the iterates $\{f^n\}_n$, and its complement, i.e., the set of points with nonstable behavior. These two complementary sets are called the *Fatou set*, and the *Julia set*, respectively. In order to understand the topology of the Julia set, the main strategy is considered to be the study of rays and their landing behaviors. Roughly speaking, dynamic rays are unbounded curves in the escaping set (i.e., the set of points which escape to infinity under iteration), formed by orbits escaping to infinity according to some symbolic dynamics. A dynamic ray $\gamma : (-\infty, \infty) \to \mathbb{C}$ for a holomorphic map f is periodic if there exists $k \in \mathbb{N}$, such that $f^k(\gamma) \subset \gamma$, and in particular, is fixed if $k = 1$. We say γ lands if $\lim_{t\to-\infty} \gamma(t)$ exists.

Similar to the analysis in the dynamical plane, the study of the parameter spaces can be considered in terms of the study of curves consisting of escaping parameters and in terms of their landing behaviors. Those rays are called parameter rays. By escaping parameter, we mean a parameter for which the singular value (i.e., a point with the property that, at least one of the inverse branches of the map is not well-defined, and not univalent in any neighborhood of it) escapes to infinity along dynamic rays with a prescribed (combinatorial) identification. We say a parameter ray $\Gamma : (-\infty, \infty) \to \mathbb{C}$ lands if $\lim_{t\to-\infty} \Gamma(t)$ exists.

There are two main questions in the study of rays: whether a ray lands, or a considered point is a landing point of a ray. In order to distinguish these two questions, we use the terms "landing" and "converse landing", respectively. It is of course reasonable to study the existence of curve structure first, and then to explore landing behaviors. To the question of the existence of rays for transcendental entire families, although computer simulations suggest that escaping parameters form curves in the parameter spaces, there is no complete answer, except for the exponential family. The first study of the existence of parameter rays in the exponential family was carried out by Devaney and coauthors in [2]. Schleicher worked broadly on combinatorial analysis of the exponential family in his habilitation thesis [11], and his student Förster completed the construction of the parameter rays in his diploma thesis [5] with a combinatorial perspective, which then appeared in [6]. How-

ever, landing properties of periodic parameter rays in the habilitation thesis is still unpublished. To our knowledge, nothing has been proved for transcendental entire families in a general setting. In this paper, we prove a converse landing theorem for curves in the parameter spaces of holomorphic families of holomorphic maps. The theorem itself already gives the existence of *a fixed parameter curve (or a fixed parameter ray piece)* in a bounded sector which lands at a parabolic parameter. By a fixed parameter curve, we mean a curve in the parameter plane, consisting of parameter values, for which the singular value is on a fixed dynamic ray with a prescribed (combinatorial) identification (for simplicity of the notation, we consider fixed rays, which we can also apply for the qth iterate of a q-periodic ray). So in some sense, this can be considered as a partial answer to the existence problem of the parameter rays, particularly in transcendental entire families.

Our proof relies on having a holomorphic motion in a subset of the escaping set. In polynomial dynamics, the required holomorphic motion is already guaranteed by the Böttcher's Theorem, so our theorem applies to all polynomials of degree $d \geq 2$. In the transcendental entire case, holomorphic motion is related to *quasiconformal equivalence*. By quasiconformal equivalence of maps f and g, it is inferred that there exist a pair of quasiconformal maps ϕ and ψ, such that

$$\psi \circ f = g \circ \phi.$$

Set for $K > 0$,

$$\mathcal{J}_K(f) := \{z \in \mathcal{J}(f) : \ \operatorname{Re} f^n(z) \geq K \ \text{ for all } \ n \geq 0\}.$$

Here we cite a result from [9], which is a consequence of the fact that quasiconformally equivalent maps with bounded singular sets (i.e., of class $\mathcal{B}$) are quasiconformally conjugate on their escaping sets.

Proposition 1.1. *[9, Prop 3.6] Let $f \in \mathcal{B}$. Suppose M is a finite-dimensional complex manifold, with base point λ_0. Suppose $\{f_\lambda\}_{\lambda \in M}$ is a family of entire functions quasiconformally equivalent to f, given by the equivalence*

$$\psi_\lambda \circ f = f_\lambda \circ \phi_\lambda,$$

where $\psi_{\lambda_0} = \phi_{\lambda_0} = Id$, and ψ_λ and ϕ_λ depend analytically on λ. Let N be a compact neighborhood of λ_0. Then there exists a constant $K > 0$, such that, for every $\lambda \in N$, there exists an injective function $H_\lambda : \mathcal{J}_K(f) \to \mathcal{J}(f_\lambda)$, satisfying:

i. $H_{\lambda_0} = Id$,

ii. $H_\lambda \circ f = f_\lambda \circ H_\lambda$, *and*

iii. *for fixed* $z \in \mathcal{J}_K(f)$, *the map* $\lambda \mapsto H_\lambda(z)$ *is analytic on the interior of* N.

2 Setup - theorem - idea of the proof

Consider a one parameter holomorphic family of holomorphic maps $\{f_a\}_{a\in\mathbb{D}(a_0,r)}$ (where $\mathbb{D}(a_0, r)$ denotes a disk with radius r and centered at a_0) which has a nonpersistent parabolic fixed point z_0 for the parameter a_0 satisfying

i. $f'_{a_0}(z_0) = 1$, and

ii. $f''_{a_0}(z_0) \neq 0$ (i.e., nondegenerate).

Suppose there exists $R > 0$, such that for all $a \in \mathbb{D}(a_0, r)$, f_a has exactly one singular value $s(a)$ in $\overline{\mathbb{D}(z_0, R)}$ which depends holomorphically on a. Suppose also that the attracting petal of the parabolic fixed point z_0 contains only $s(a_0)$ of all singular values (if there are more).

Possibly reparametrizing by a map $\lambda \mapsto a(\lambda)$, which is either univalent, or a degree 2 cover branched above a_0, and changing the coordinate in a holomorphically depending way, we can assume f_a takes the form

$$f_{a(\lambda)}(z) = (1 + 2\lambda^n)z + z^2 + ... \tag{1}$$

near 0. Obtaining this form is the subject of Section 3.

For a forward invariant curve $\gamma_{a_0} : (-\infty, \mathcal{T}) \to \mathbb{C}$, $\mathcal{T} \in (-\infty, \infty]$, $t \mapsto \gamma_{a_0}(t)$, we shall suppose it is parametrized such that for $t < \mathcal{T} - 1$, $f_{a_0}(\gamma_{a_0}(t)) = \gamma_{a_0}(t+1)$. In order to avoid confusion, we call the parameter t of a ray, "the potential".

Main Theorem. *Let* $\{f_a\}_{a\in\mathbb{D}(a_0,r)}$ *be as is stated above. Suppose there is a forward invariant curve* $\gamma_{a_0} : (-\infty, \mathcal{T}) \to \mathbb{C}$, $t \mapsto \gamma_{a_0}(t)$ *such that* $\lim_{t\to-\infty} \gamma_{a_0}(t) = z_0$, *and that* γ_{a_0} *does not contain any critical points (i.e., image of a branch point of* f_{a_0}*). Suppose also that there exists* $T > 0$,

$T+1<\mathcal{T}$, a neighborhood $\mathbb{D}(a_0,\delta)$ of a_0, such that $\delta=\delta(T)$, $\delta<r$, and an equivariant holomorphic motion H, i.e.,

$$\begin{aligned} H:\mathbb{D}(a_0,\delta)\times\gamma_{a_0}[T,\mathcal{T}) &\to \widehat{\mathbb{C}} \\ (a,\gamma_{a_0}(t)) &\mapsto \gamma_a(t) \end{aligned}$$

satisfying

$$f_a\Big(H(a,\gamma_{a_0}(t))\Big)=\gamma_a(t+1).$$

Then for some $t_0\in\mathbb{R}$ with $t_0+1<T$, there exists a simple curve $\Gamma:(-\infty,t_0]\to\mathbb{D}(a_0,\delta)$ in the parameter plane, such that for each point $a=\Gamma(t)$: $s(a)=\gamma_a(t)$. Moreover,

$$\Gamma(t)\to a_0 \text{ as } t\to-\infty.$$

The reader unfamiliar with the Fatou coordinates and parabolic implosion is recommended to see Section 4 for the notations and the basic concepts we use to explain the idea of the proof.

Suppose γ_{a_0} is a forward invariant curve landing at the parabolic fixed point z_0. This curve lands through the repelling petal Ω^-. Under the outgoing Fatou coordinate $\phi^-_{a_0}$, $\gamma_{a_0}\cap\Omega^-$ maps to a 1-periodic curve, which extends 1-periodically to $-\infty$ and $+\infty$, say $\widetilde{\gamma}_{a_0}$. We take a parameter a nearby, for which the holomorphic motion $H(a,\gamma_{a_0}[t',\mathcal{T}))=\gamma_a[t',\mathcal{T})$ and the Douady-Fatou coordinates $\phi^\pm_a$ are well defined. The curve $\gamma_a[t',\mathcal{T})\cap\Omega^-$ maps to a 1-periodic curve under the outgoing Douady-Fatou coordinate ϕ^-_a which extends 1-periodically to $-\infty$ and $+\infty$, say $\widetilde{\gamma}_a$. The 1-periodicity of the curves $\widetilde{\gamma}_{a_0}$ and $\widetilde{\gamma}_a$ transfers the relationship coming from the continuity to small potentials. After change of parameters and suitable normalizations, we compare $\widetilde{\gamma}_a(t)$ and $\phi^-_{a_0}(s(a))$ and see that, there exists a "central" sector with corner point at a_0, such that given sufficiently small potential t_0, for all $t\le t_0$, there is a unique parameter a' in this sector, which satisfies

$$s(a')=\gamma_{a'}(t),$$

applying Rouché's Theorem. What we mean by central sector is going to be explained in Section 6.

This relation induces a map Γ which assigns a unique parameter a' to potential $t \le t_0$. We parametrize Γ so that, when $s(a') = \gamma_{a'}(t)$, we write $a' = \Gamma(t)$. By using a continuity argument, we show that Γ defines a curve. Moreover our construction shows as $t \to -\infty$, $\Gamma(t) \to a_0$.

Here we want to emphasize that, for changing parameter values a, the parabolic fixed point z_0 must have a neighborhood, say a disk $\mathbb{D}(z_0, R)$, where the only singular value is $s(a)$. Moreover, the attracting petal must have exactly one singular value. Otherwise, the orbit of another singular value may interfere. We require holomorphic dependence of the singular value $s(a)$ as a function of the parameter for the application of Rouché's Theorem.

Let us give the structure of this article. The proof of Main Theorem requires changing parameters twice. The first one is the subject of the next section, where we obtain the form given by (1). Then we show the existence of sectors in the parameter plane, where the multiplier of the fixed point 0 is univalent. In Section 4 we recall the Fatou coordinates and the parabolic implosion phenomenon, and introduce some notations. In Section 5 we perform another change of parameters using the Douady-Fatou coordinates. In Section 6, we collect all the information and prove Main Theorem. The last section is devoted to applying Main Theorem to transcendental entire families.

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3 Reparametrization-I

Consider a one parameter holomorphic family of holomorphic maps $\{f_a\}_{a \in \mathbb{D}(a_0, r)}$, which has a nondegenerate parabolic fixed point at z_0 with multiplier 1 for the parameter a_0. When the map f_{a_0} is perturbed to f_a nearby, if z_0 is not

a persistent parabolic fixed point, it bifurcates into two distinct fixed points, say $z_1(a)$ and $z_2(a)$. Since the family $\{f_a\}_{a\in\mathbb{D}(a_0,r)}$ depends holomorphically on the parameter, the multipliers of these two fixed points depend holomorphically on the parameter. In this section, we follow one of the fixed points as a function of the parameter. We shall show that, there exists a sector in the parameter plane, on which the multiplier of that fixed point depends univalently on the parameter. This result is stated in Proposition 3.2. To start with, we change the coordinate in the dynamical plane, so that we have a simpler form to deal with.

Lemma 3.1. *Let $\{f_a\}_{a\in\mathbb{D}(a_0,r)}$ be a one parameter holomorphic family of holomorphic maps, which has a nondegenerate parabolic fixed point at z_0 with multiplier 1 for the parameter a_0. Suppose $\mathbb{D}(z_0,R)$ is a Euclidian disk in the dynamical plane for f_{a_0}, such that f_{a_0} has only z_0 as fixed point in $\overline{\mathbb{D}(z_0,R)}$. Then, there exists $r'>0$, ($r'<r$) such that for all a in $\mathbb{D}(a_0,r')$, f_a has two fixed points in $\mathbb{D}(z_0,R)$, counted with multiplicity.*

Proof. Since f_{a_0} has a nondegenerate parabolic fixed point at z_0, the multiplicity of z_0 is 2. Let $\delta := \inf_{|z-z_0|=R}|f_{a_0}(z)-z|$. Since f_a depends continuously on the parameter, there exists $r'>0$ such that for all $a\in\mathbb{D}(a_0,r')$ and for all z such that $|z-z_0|=R$,

$$|f_a(z)-f_{a_0}(z)|<\delta.$$

By Rouché's Theorem, $f_{a_0}(z)-z$ and $f_a(z)-z$ have the same number of zeroes in $\mathbb{D}(z_0,R)$, counted with multiplicity. Hence f_a has two zeroes in $\mathbb{D}(z_0,R)$, counted with multiplicity.

□

Lemma 3.1 gives the existence of two fixed points, $z_1(a)$ and $z_2(a)$, with the possibility that they are equal. Possibly taking $r'<r$, and assuming z_0 is nonpersistent, we can suppose that $z_1(a)\neq z_2(a)$ in $\mathbb{D}(0,r')$. Let us concentrate on one of the fixed points, say $z_1(a)$ and its multiplier $\mu_1(a)$.

Proposition 3.2. *Let $\{f_a\}_{a\in\mathbb{D}(a_0,r)}$ be a one parameter holomorphic family of holomorphic maps, which has a nondegenerate parabolic fixed point at z_0 with multiplier 1 for the parameter a_0. If z_0 is not a persistent parabolic fixed point, there exists an open sector Δ in the parameter plane with corner point at a_0, such that the multiplier map $\mu_1:\Delta\to\mathbb{C}$ of the fixed point $z_1(a)$ is univalent.*

The proof requires change of coordinates both in the parameter plane and in the dynamical plane. First observe the following:

Lemma 3.3. *Let $\{z_1(a), z_2(a)\}$ be the set of fixed points of f_a in $\mathbb{D}(z_0, R)$ as was given in Lemma 3.1. There exists an affine change of coordinate in the dynamical plane, which depends holomorphically on the parameter such that the corresponding fixed points are symmetric with respect to* 0*.*

Proof. Define $\mathbb{S}(z_0, R) := \partial\mathbb{D}(z_0, R)$. The sum of the fixed points $z_1(a)$ and $z_2(a)$ is the holomorphic function, given by

$$a \mapsto \sigma(a) = \frac{1}{2\pi i} \oint_{\mathbb{S}(z_0,R)} \frac{w(f'_a(w) - 1)}{f_a(w) - w} dw.$$

The linear map $M_a(z) = z - \frac{1}{2}\sigma(a)$ conjugates f_a to the holomorphic map which has two symmetric fixed points with respect to 0. $\square$

After the affine change of coordinate given in Lemma 3.3, let us denote by $z(a)$ and $-z(a)$, the symmetric fixed points with respect to 0, which correspond to $z_1(a)$ and $z_2(a)$. It is possible to find a local expression for them. This is given by the following lemma.

Lemma 3.4. *Consider a map which has two symmetric fixed points $\pm z(a)$, as given by Lemma 3.3. Taking a covering of degree at most* 2 *in the parameter plane, the fixed points are holomorphic functions of the parameter of the form:*

$$a \mapsto \pm z(a) = \pm(a - a_0)^n + O\Big((a - a_0)^{n+1}\Big). \tag{2}$$

Proof. Possibly reducing r', we may follow the fixed point $z(a)$ analytically along any path in $\mathbb{D}^*(a_0, r')$. Consider a simple closed curve $\gamma : [0, 1] \to \mathbb{C}$ around a_0 in $\mathbb{D}^*(a_0, r')$ in the parameter plane. For the parameters $\gamma(0)$ and $\gamma(1)$, we observe two different situations in the corresponding dynamical planes. Either

i. the locations of the fixed points $z(a)$ and $-z(a)$ are the same for $\gamma(0)$ and $\gamma(1)$, or,

ii. they have interchanged their locations.

In case i., $z(a)$ is a holomorphic function of $a \in \mathbb{D}^*(a_0, r)$ of local degree $n \geq 1$. Then (2) is obtained after scaling. In case ii., we consider a branched

covering map of degree 2 from a domain $\widehat{\Omega}$ to $\mathbb{D}(a_0, r)$ in the parameter plane (eg., $b \mapsto a(b) = a_0 + b^2$),

$$\begin{array}{rcl} \xi : \widehat{\Omega} & \to & \mathbb{C}, \\ b & \mapsto & \xi(b) = a, \;\; \xi(b_0) = a_0. \end{array}$$

Hence the map $b \mapsto z(\xi(b))$ is a branched covering of degree $n \geq 1$. In this new paramerization we are back in i. So, the fixed points have the form:

$$b \mapsto \pm(b - b_0)^n + O((b - b_0)^{n+1}),$$

after scaling. $\square$

In the following lemma, we find the multiplier $\mu(a)$ of the fixed point $z(a)$ given in (2).

Lemma 3.5. *Possibly changing the coordinate in the parameter plane, the multiplier of the fixed point $z(a)$ can be given by*

$$\lambda \mapsto \mu_\lambda := 1 + 2\lambda^n. \tag{3}$$

Proof. After the change of coordinates given in Lemma 3.3, f_a can be written as

$$f_a(z) = z + (z^2 - z(a)^2)h_a(z)$$

near 0, where $h_a : \mathbb{D}(z_0, R) \to \mathbb{C}$ is a nonvanishing holomorphic map. The multiplier μ at $z(a)$ is equal to

$$\mu(a) = 1 + 2z(a)h_a(z(a)).$$

Substituting $z(a) = (a - a_0)^n + O((a - a_0)^{n+1})$ (by Lemma 3.4), we have

$$\mu(a) = 1 + 2(a - a_0)^n F(a),$$

where F is a holomorphic map, with $F(a_0) \neq 0$. Hence F has a well defined nth root around a_0. So we can write

$$\begin{aligned} \mu(a) &= 1 + 2(a - a_0)^n \big((F(a))^{1/n}\big)^n \\ &= 1 + 2\big((a - a_0)(F(a))^{1/n}\big)^n. \end{aligned} \tag{4}$$

We define $\lambda(a) := (a - a_0)(F(a))^{1/n}$. Since $F(a_0) \neq 0$, then $\lambda'(a_0) \neq 0$, and hence there exists a neighborhood of a_0, where the map $a \mapsto \lambda(a)$ is univalent. Thus we can consider λ as a new parameter. Rewriting (4) in the new parameter, the multiplier map $\mu(a)$ of $z(a)$ is written as $\lambda \mapsto 1+2\lambda^n$. $\square$

Now, it is easy to see that by an affine change of coordinate in the dynamical plane, the fixed point $z(a)$ translates to the origin, and we obtain the form given by (1). With this result, we are ready to prove Proposition 3.2.

of Proposition 3.2. In the form (1), let us denote the multiplier map of 0 by μ_λ, which is of the form (3) given in Lemma 3.5. In the λ-parameter plane, consider the restriction of the domain of the multiplier map μ_λ to a sector

$$\Lambda = \{\lambda, \;\; 2\pi\theta_1 < \arg(\lambda) < 2\pi\theta_2, \;\; \theta_1, \theta_2 \in (0,1)\}. \tag{5}$$

Then $\arg(\mu_\lambda - 1) \in (2n\pi\theta_1, 2n\pi\theta_2)$. Provided that $n(\theta_2 - \theta_1) < 1$, μ_λ is a univalent map in Λ. $\square$

4 Fatou coordinates and parabolic implosion

Here we give a summary of the phenomenon of the Fatou coordinates and the parabolic implosion, mostly out of [7], [8] and [12].

Consider the holomorphic map f_{a_0}, which has a parabolic fixed point at $z = 0$, with

i. $f'_{a_0}(0) = 1$, and

ii. $f''_{a_0}(0) \neq 0$.

Under these assumptions, the map f_{a_0} can be written as

$$f_{a_0}(z) = z + z^2 + ...$$

near 0. The parabolic fixed point 0 is on the boundary of its immediate parabolic basin. Using $I(z) := -\frac{1}{z}$ define

$$f^*_{a_0}(z) := I \circ f_{a_0} \circ I^{-1}(z) = z + 1 + \frac{a^*}{z} + O(\frac{1}{z^2}), \quad a^* \in \mathbb{C}. \tag{6}$$

The map $f^*_{a_0}$ can be seen as an approximation to the translation $z \mapsto z+1$. Because of this, I is often called *a pre-Fatou coordinate*. Set for large $L > 0$

$$\begin{aligned} \Omega_- &:= \{z \in \mathbb{C};\ \operatorname{Re} z < -L + |\operatorname{Im} z|\} \text{ and} \\ \Omega_+ &:= \{z \in \mathbb{C};\ \operatorname{Re} z > L - |\operatorname{Im} z|\}. \end{aligned}$$

so that on $\overline{\Omega_-} \cup \overline{\Omega_+}$, the map $f^*_{a_0}$ is injective, and satisfies

$$f^*_{a_0}(\overline{\Omega_+}) \subset \Omega_+ \cup \{\infty\} \text{ and } \overline{\Omega_-} \subset f^*_{a_0}(\Omega_- \cup \{\infty\}).$$

Attracting and repelling petals are defined by $\Omega^+ := I^{-1}(\Omega_+)$ and $\Omega^- := I^{-1}(\Omega_-)$, respectively. The union $\Omega^- \cup \Omega^+ \cup \{0\}$ is simply connected.

Theorem 4.1. *(Existence of the Fatou coordinates) There exists injective holomorphic mappings, $\phi^{a_0}_- : \Omega_- \to \mathbb{C}$ and $\phi^{a_0}_+ : \Omega_+ \to \mathbb{C}$, satisfying*

$$\phi^{a_0}_-(f^*_{a_0}(z)) = \phi^{a_0}_-(z) + 1 \text{ and } \phi^{a_0}_+(f^*_{a_0}(z)) = \phi^{a_0}_+(z) + 1.$$

Moreover,

i. $\phi^{a_0}_+$ and $\phi^{a_0}_-$ are unique up to addition of constants.

ii. If $f^*_{a_0}$ is as in (6), then

$$\phi^{a_0}_\pm(z) = z - a^* \log z + c_\pm + o(1), \quad c_+, c_- \in \mathbb{C} \tag{7}$$

as $z \to \infty$.

For the proof of Theorem 4.1, see for example, [12, Prop 2.2.1].

Definition 4.2. We define *the incoming Fatou coordinate* by

$$\phi^+_{a_0} := \phi^{a_0}_+ \circ I : \Omega^+ \to \mathbb{C},$$

and *the outgoing Fatou coordinate* by

$$\phi^-_{a_0} := \phi^{a_0}_- \circ I : \Omega^- \to \mathbb{C}.$$

In the dynamical plane for f_{a_0}, we observe croissant shaped fundamental domains $P^+_{a_0}$, and $P^-_{a_0}$, each of which is bounded by a pair of curves meeting at 0, where one curve maps to the other under f_{a_0} (see Figure 1). These fundamental domains may overlap around 0.

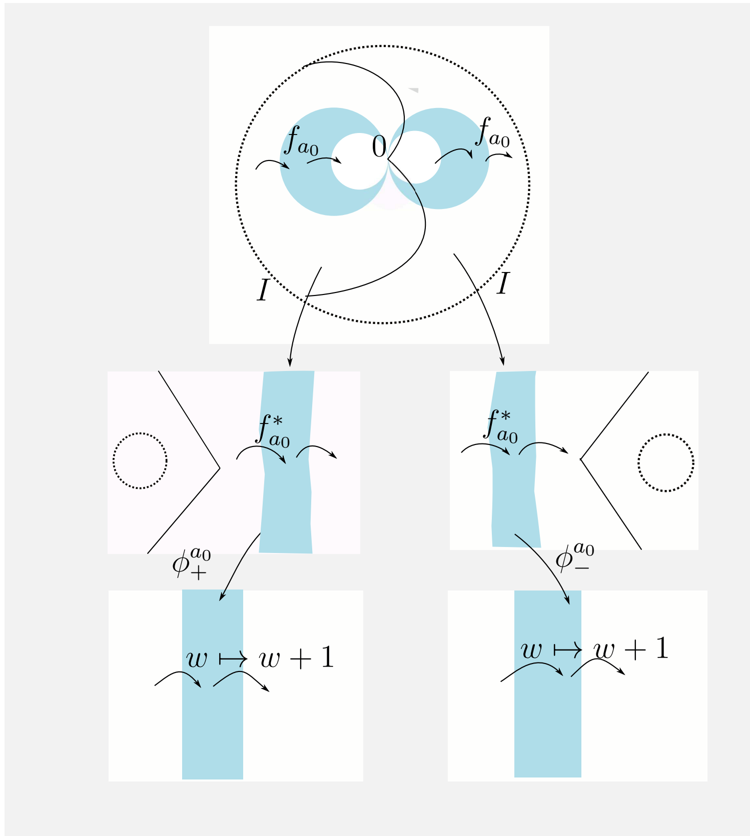

Figure 1: Fatou Coordinates.

Remark 4.3. The incoming Fatou coordinate $\phi^+_{a_0}$ is univalent in Ω^+, and the outgoing Fatou coordinate $\phi^-_{a_0}$ is univalent in Ω^-. Let us define $\psi^+_{a_0} := (\phi^+_{a_0})^{-1}$, and $\psi^-_{a_0} := (\phi^-_{a_0})^{-1}$. By using the dynamics, $\phi^+_{a_0}$ extends to the whole parabolic basin, and $\psi^-_{a_0}$ extends to the whole complex plane. However, these extensions are not univalent anymore because of the presence of singular value(s).

Definition 4.4. Let Ω^+ and Ω^- be the attracting and repelling petals, such that $\Omega^+ \cap \Omega^- \neq \emptyset$. Suppose $U^u_{a_0}$ and $U^l_{a_0}$ are the connected components of $\Omega^+ \cap \Omega^-$. We call the domains $S^u_{a_0} = \bigcup_{i\in\mathbb{Z}} f^i_{a_0}(U^u_{a_0})$ and $S^l_{a_0} = \bigcup_{i\in\mathbb{Z}} f^i_{a_0}(U^l_{a_0})$, *the upper and the lower sepal*, respectively.

Definition 4.5. Let $S^u_{a_0}$ and $S^l_{a_0}$ be upper and lower sepals, and let $V^{u,l}_{a_0} := (\psi^-_{a_0})^{-1}(S^{u,l}_{a_0})$ and $W^{u,l}_{a_0} := \phi^+_{a_0}(S^{u,l}_{a_0})$. We denote the restriction of $\phi^+_{a_0} \circ \psi^-_{a_0}$ to

$V_{a_0}^u$ by

$$H_{a_0}^u : V_{a_0}^u \to W_{a_0}^u,$$

and the restriction of $\phi_{a_0}^+ \circ \psi_{a_0}^-$ to $V_{a_0}^l$ by

$$H_{a_0}^l : V_{a_0}^l \to W_{a_0}^l.$$

The maps $H_{a_0}^u$ and $H_{a_0}^l$ are called *the upper and the lower lifted Horn map*, respectively.

By construction

$$H_{a_0}^{u,l} \circ T_1 = T_1 \circ H_{a_0}^{u,l}.$$

Since the Fatou coordinates are uniquely defined up to translation, the lifted Horn maps are uniquely defined up to pre and post composition by translations. Indeed, suppose the pairs $\phi_{a_0}^\pm$ and $\widetilde{\phi}_{a_0}^\pm$ are Fatou coordinates of the same map. Then, there exist constants $c, k \in \mathbb{C}$, and translations $T_c : z \mapsto z + c$, and $T_k : z \mapsto z + k$, such that $\phi_{a_0}^- = T_c \circ \widetilde{\phi}_{a_0}^-$, and $\phi_{a_0}^+ = T_k \circ \widetilde{\phi}_{a_0}^+$. Define $\widetilde{\psi}_{a_0}^- := (\widetilde{\phi}_{a_0}^-)^{-1}$. The corresponding lifted Horn maps $H_{a_0}^{l,u} = \phi_{a_0}^+ \circ \psi_{a_0}^-$ and $\widetilde{H}_{a_0}^{l,u} = \widetilde{\phi}_{a_0}^+ \circ \widetilde{\phi}_{a_0}^-$ satisfy

$$\begin{aligned} H^{l,u} &= \phi_{a_0}^+ \circ (\phi_{a_0}^-)^{-1} \\ &= T_k \circ \widetilde{\phi}_{a_0}^+ \circ (T_c \circ \widetilde{\phi}_{a_0}^-)^{-1} \\ &= T_k \circ \widetilde{\phi}_{a_0}^+ \circ \widetilde{\psi}_{a_0}^- \circ T_{-c} \\ &= T_k \circ \widetilde{H}_{a_0}^{l,u} \circ T_{-c}. \end{aligned} \tag{8}$$

With this property, we are free to choose normalizations of $\phi_{a_0}^\pm$ such that the upper lifted Horn map is of the form

$$H_{a_0}^u(z) = z + o(z^2). \tag{9}$$

In this case the lower lifted Horn map is given by

$$H_{a_0}^l(z) = z - 2\pi i a^* + o(z^2),$$

(see, for example [3], or [7, Chpt 4], for *formal invariant*). We shall work with the normalized form (9).

We are interested in perturbations of the map $f_{a_0} : z \mapsto z + z^2 + O(z^3)$ to a map f_a nearby. In the nondegenerate case, in general, the parabolic fixed

point bifurcates into two fixed points, each of which has multiplier close to 1. These fixed points can be attracting, repelling, or indifferent. We are going to consider the bifurcation, where the multipliers of the fixed points are not real. In such bifurcations, the fundamental domains continue to exist and the boundary curves of each fundamental domain meet at the two distinct fixed points. We denote the fundamental domains for f_a by P_a^+ and P_a^-. A gate opens between these two fixed points, and any point in P_a^+ passes under the dynamics through the gate and eventually enters P_a^-. However, it needs a very large number of iterates to pass from P_a^+ to P_a^-. This is referred to as "egg beater" dynamics (see Figure 2). More precisely:
Theorem 4.6. *(Existence of Douady-Fatou coodinates [12, Prop 3.2.2]) Consider the holomorphic family of holomorphic maps* $\{f_a\}_{a\in\mathbb{D}(a_0,r)}$, *such that*

$$f_{a_0}(z) = z + z^2 + ...$$

near 0. *There exists an open sector* Δ *in the parameter plane with corner point* a_0, *such that for all* $a \in \Delta$, *there exists a pair of conformal maps* ϕ_a^+ *and* ϕ_a^-, *which conjugate the dynamics to the translation* $z \mapsto z+1$. *After a suitable normalization,* ϕ_a^+ *and* ϕ_a^- *depend holomorphically on* a *(continuously on* a *on* $\overline{\Delta}$*). Moreover, the difference* $\phi_a^-(z) - \phi_a^+(z)$ *is constant for fixed parameter* a.

Let us explain Theorem 4.6 with the results we have got so far. When we consider the form given by (1), one of the fixed points stays put at 0 under the perturbation. Its multiplier $\mu_\lambda = 1 + 2\lambda^n$ for some $n \in \mathbb{N}$ is given in Lemma 3.5. Suppose for example, μ_λ satisfies $|\arg(\mu_\lambda - 1)| < \frac{\pi}{4}$. Then Theorem 4.6 implies that $\phi_a^\pm \rightrightarrows \phi_{a_0}^\pm$ on compact sets in the petals $\Omega^\pm$. We call ϕ_a^+ and ϕ_a^-, *the incoming and the outgoing Douady-Fatou coordinates*, respectively. The lifted Horn maps are defined in the same way as in the parabolic situation, and we denote them by H_a^u and H_a^l. Note that $H_a^u \rightrightarrows H_{a_0}^u$ on compact sets in some upper half plane, and the domain of convergence can be extended to $\operatorname{Im} z \to +\infty$. Similarly $H_a^l \rightrightarrows H_{a_0}^l$ on compact sets in some lower half plane, and the domain of convergence can be extended to $\operatorname{Im} z \to -\infty$.

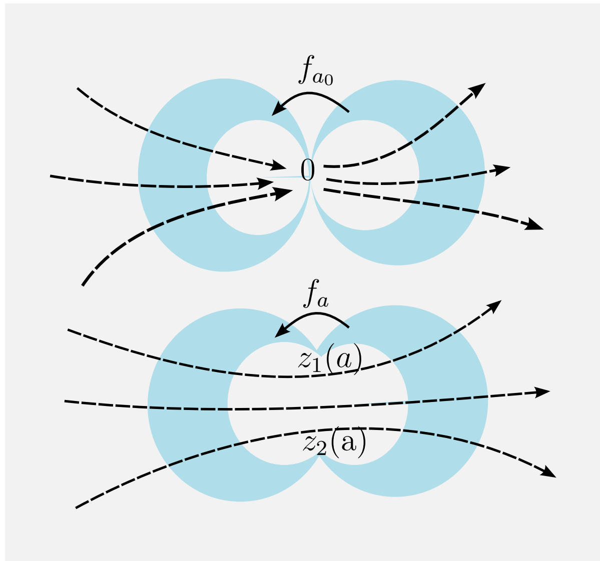

Figure 2: Bifurcation of the parabolic fixed point.

5 Reparametrization-II

Recall that by Theorem 4.6, the difference $\phi_a^-(z) - \phi_a^+(z)$ is constant for each fixed parameter. Define

$$B(a) := \phi_a^- - \phi_a^+. \tag{10}$$

Observe that $\operatorname{Re} B(a) < 0$. Since after suitable normalizations, ϕ_a^+ and ϕ_a^- depend holomorphically on the parameter on some open sector, B is also holomorphic on the same sector. In this section, our aim is to show that the sector can be chosen so that $a \mapsto B(a)$ is univalent. More precisely, we will prove the following.

Theorem 5.1. *Under the hypothesis of Theorem 4.6 and after suitable normalizations of $\phi_a^\pm$, there exists an open sector Δ, such that the map $B : \Delta \to \mathbb{C}$ given by (10) is univalent with respect to the parameter a.*

The strategy we are going to use is to relate B with the multiplier map μ_λ given by (3) in Lemma 3.5. We will present a general case in Subsections 5.1, 5.2 and 5.3. The relationship between B and μ_λ is given in Proposition 5.5, as a consequence.

5.1 Constructing a Riemann surface

Consider a map H which is defined in some simply connected domain $\mathcal{H}$ containing an upper half plane such that

$$H(z) = z + \beta + o(1) \text{ as } \operatorname{Im} z \to \infty$$

with $\operatorname{Re}\beta < 0$. By taking a subset of $\mathcal{H}$, we will construct a Riemann surface with the iterates of H the transition maps.

Take a domain $\mathcal{M} := \{z; \quad \operatorname{Im} z > m\}$ such that $\mathcal{M} \subset \mathcal{H}$. We assume that for all $z \in \mathcal{M}$

$$|H(z) - z - \beta| < \frac{|\beta|}{4} \quad \text{and} \tag{11}$$

$$|H'(z) - 1| < \frac{1}{4}. \tag{12}$$

One can see that H is univalent in $\mathcal{M}$ by (11).

Take $z_0 = x_0 + iy_0 \in \mathcal{M}$ such that $H(z_0) \in \mathcal{M}$. Set

$$\begin{aligned}
\gamma_0 &:= \{z; \ \operatorname{Re} z = x_0, \ \operatorname{Im} z \in [y_0, \infty)\}, \\
\gamma_1 &:= \{z; \ \frac{\operatorname{Re}(z - z_0)}{\operatorname{Re}(H(z_0) - z_0)} = \frac{\operatorname{Im}(z - z_0)}{\operatorname{Im}(H(z_0) - z_0)}, \ \operatorname{Re} H(z_0) \le \operatorname{Re} z \le \operatorname{Re} z_0\}, \\
\gamma_2 &:= H(\gamma_0).
\end{aligned}$$

such that $\gamma_0 \cap \gamma_2 = \emptyset$, $\gamma_1 \cap \gamma_2 = \emptyset$. Here γ_1 is the line segment connecting z_0 to $H(z_0)$. Denote by $\mathcal{S}$, the region in $\mathcal{M}$ which is bounded by the curve $\gamma_0 \cup \gamma_1 \cup \gamma_2$. Notice that the iterates of a point in $\mathcal{M}$ under H have at most one point in $\mathcal{S}$.

Claim 5.2. *For all $z \in \mathcal{H}$, $|\arg \frac{H(z)-z}{\beta}| < \frac{\pi}{12}$.*

Proof. In Figure 3, it is shown that $|\arg \frac{H(z)-z}{\beta}|$ takes its maximum at the arguments of the angles $\widehat{BOA}$ and $\widehat{EOB}$. We use the principle that for $\frac{5\pi}{12} - \frac{\pi}{12} - \frac{\pi}{2}$ triangle, the height to the hypotenuse is a quarter of the length of the hypotenuse. We compare the triangle BOA with a $\frac{5\pi}{12} - \frac{\pi}{12} - \frac{\pi}{2}$ triangle. By hypothesis, we have $|AB| = \frac{|OB|}{4}$. Since $|AD| < |AB|$, $|AD| < \frac{|OB|}{4}$ and hence $\arg(\widehat{BOA}) < \frac{\pi}{12}$.

□

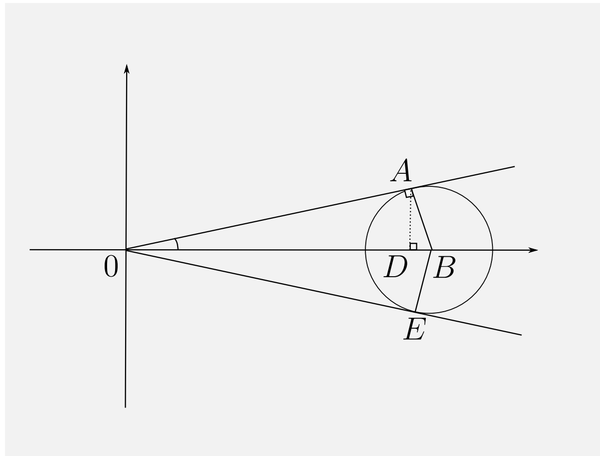

Figure 3: $\frac{H(z)-z}{\beta}$-plane.

Notice that since $\operatorname{Re}\beta < 0$, $\frac{\pi}{2} < \arg\beta < \frac{3\pi}{2}$. By the observation above, $|\arg(H(z) - z) - \arg\beta| < \frac{\pi}{12}$, i.e.,

$$-\frac{\pi}{12} + \arg\beta < \arg(H(z) - z) < \frac{\pi}{12} + \arg\beta.$$

Consequently

$$-\pi - \frac{\pi}{12} + \arg\beta < \arg(z - H(z)) < -\pi + \frac{\pi}{12} + \arg\beta.$$

Set $\theta_1 := \max\{0, -\pi + \frac{\pi}{12} + \arg\beta\}$, and $\theta_2 := \min\{\pi, -\frac{\pi}{12} + \arg\beta\}$. Let l_1, l_2 be the half lines parametrized by $[0, \infty)$, given as

$$\begin{aligned} l_1(t) &:= z_0 + te^{i\theta_1}, \\ l_2(t) &:= H(z_0) + te^{i\theta_2}. \end{aligned}$$

Then the complex plane is separated into two unbounded regions, say $\mathcal{R}$ and $\mathcal{R}^c$, by the curve $l_1 \cup \gamma_1 \cup l_2$. We denote the one containing $\mathcal{S}$ by $\mathcal{R}$ (see Figure 4). Note that for all points $z \in \mathcal{R}$ if $H^n(z) \in \mathcal{R}$, then $H(z), H^2(z), ..., H^{n-1}(z)$ are also in $\mathcal{R}$. Define an equivalence relation $\backsim_H$ in $\mathcal{R}$, by defining $z \backsim_H w$ if and only if there exists $n \in \mathbb{N}$ such that either $z = H^n(w)$, or $w = H^n(z)$.

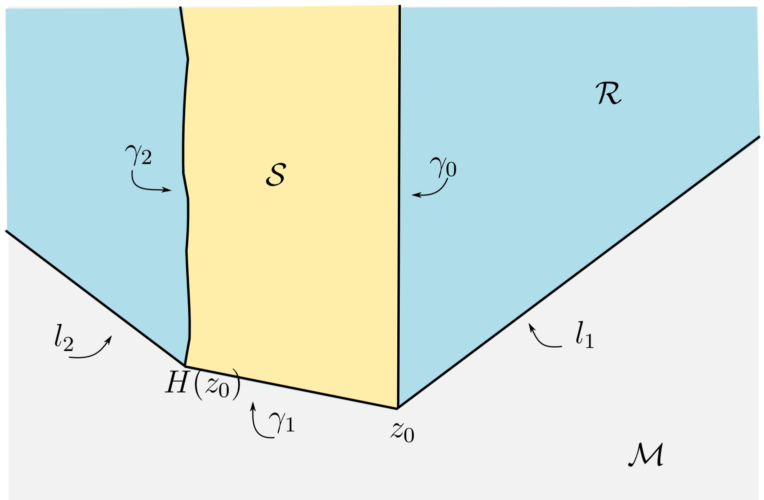

Figure 4: $\mathcal{R}$ is the region bounded by the curve $l_1 \cup \gamma_1 \cup l_2$ in $\mathcal{M}$.

Let $\mathcal{C}$ denote the quotient space $\mathcal{R}/\backsim$, and $\pi : \mathcal{R} \to \mathcal{C}$ denote the natural projection. We equip $\mathcal{C}$ with the quotient topology. The quotient topology is Hausdorff. In order to see that, it is enough to consider two representatives w_1 and w_2 in $\mathcal{S}$ of the equivalence classes $[w_1]$ and $[w_2]$. Take ϵ neighborhoods $\mathbb{D}(w_1,\epsilon), \mathbb{D}(w_2,\epsilon), \mathbb{D}(H(w_1),\epsilon), \mathbb{D}(H(w_2),\epsilon), \mathbb{D}(H^{-1}(w_1),\epsilon)$, $\mathbb{D}(H^{-1}(w_2),\epsilon)$, and
choose ϵ small enough so that they are all disjoint. Since H is well defined,

$$H^{-n}(\mathbb{D}(w_1,\epsilon)) \cap H^{-n}(\mathbb{D}(w_2,\epsilon)) = \emptyset \ \text{ for } \ n \in \mathbb{N}.$$

Moreover, $H^n(\mathbb{D}(w_1,\epsilon)) \cap H^n(\mathbb{D}(w_2,\epsilon)) = \emptyset$ for $n \in \mathbb{N}$ since H is univalent (see (12)). This shows $\mathcal{C}$ is a Hausdorff space.

We equip $\mathcal{C}$ with the unique complex structure. Then $\mathcal{C}$ is naturally a Riemann surface. Note that $\mathcal{C}$ is isomorphic to a punctured disk, since it has only one puncture which is at ∞. In particular $\mathcal{C}$ is isomorphic to $\mathbb{D}^*$ (this is explained in Subsection 5.3). We would like to point out that this construction does not give a maximal Riemann surface one can obtain out of $\mathcal{H}$, but it is sufficient to serve our needs.

Let $\sigma : \mathcal{C} \to \mathbb{C}^*$ be an isomorphism onto its image with $\sigma \circ \pi(z) \to 0$ as $\operatorname{Im} z \to \infty$. Suppose H commutes with the translation $T_1(z) = z + 1$, i.e.,

$H \circ T_1 = T_1 \circ H$. Then the translation T_1 induces a univalent map

$$\xi : \mathbb{D}^*(0,r) \to \mathbb{C}^*, \quad \text{for some } 0 < r < 1,$$

with the property that

$$\xi \circ \sigma \circ \pi = \sigma \circ \pi \circ T_1 \quad \text{on } (\sigma \circ \pi)^{-1}(\mathbb{D}^*(0,r)).$$

By the Removable Singularity Theorem, ξ extends to 0 with $\xi(0) = 0$ and $\xi'(0) \neq 0$.

As a summary, for a given map H which is defined in a simply connected domain containing an upper half plane such that $H(z) = z + \beta + o(1)$ as $\operatorname{Im} z \to \infty$, with $\operatorname{Re} \beta < 0$, and which commutes with $T_1(z) = z + 1$, we formalize the process of obtaining ξ by the 6-tuple elements of the construction

$$(z_0, \mathcal{R}, \pi, \mathcal{C}, \sigma, \xi).$$

In the next subsection we will show that the multiplier $\xi'(0)$ is independent of the construction.

5.2 Independence of the multiplier

Lemma 5.3. *Consider the by T_1 induced maps*

$$\xi_1 : \mathbb{D}(0,r_1) \to \mathbb{C}^* \quad \text{and} \quad \xi_2 : \mathbb{D}(0,r_2) \to \mathbb{C}^*$$

of the respective constructions $(z_1, \mathcal{R}_1, \pi_1, \mathcal{C}_1, \sigma_1, \xi_1)$ and $(z_2, \mathcal{R}_2, \pi_2, \mathcal{C}_2, \sigma_2, \xi_2)$. There exists a neighborhood $\mathbb{D}(0,r)$, $r \leq r_1$ where the map $\sigma_2 \circ \sigma_1^{-1}$ conjugates ξ_1 to ξ_2.

Proof. Suppose $\mathcal{R}_2 \subset \mathcal{R}_1$. Then $\pi_1|_{\mathcal{R}_2} = \pi_2$, so $\mathcal{C}_2$ is naturally a subcylinder, i.e., $\mathcal{C}_2 \subset \mathcal{C}_1$. So the isomorphisms satisfy $\sigma_2 = \sigma_1|_{\mathcal{C}_2}$. Therefore, ξ_2 is actually a restriction of ξ_1, i.e., $\xi_2 = \xi_1|_{\mathbb{D}(0,r_2)}$.

Now suppose $\mathcal{R}_1 \backslash \mathcal{R}_2 \neq \emptyset$ and $\mathcal{R}_2 \backslash \mathcal{R}_1 \neq \emptyset$. Then there always exists a construction $(z_3, \mathcal{R}_3, \pi_3, \mathcal{C}_3, \sigma_3, \xi_3)$ such that $\mathcal{R}_3 \subset (\mathcal{R}_1 \cap \mathcal{R}_2)$. The restrictions $\pi_1|_{\mathcal{R}_3} = \pi_2|_{\mathcal{R}_3} = \pi_3$ determine the cylinder $\mathcal{C}_3$. Via the isomorphisms $\sigma_3 = \sigma_1|_{\mathcal{C}_3}$ and $\widetilde{\sigma}_3 = \sigma_2|_{\mathcal{C}_3}$ the translation T_1 induces univalent maps $\xi_3 :=$

$\xi_1|_{\mathbb{D}(0,r_3)} : \mathbb{D}(0,r_3) \to \mathbb{C}^*$ and $\widetilde{\xi}_3 := \xi_2|_{\mathbb{D}(0,\widetilde{r}_3)} : \mathbb{D}(0,\widetilde{r}_3) \to \mathbb{C}^*$. Define a conformal isomorphism $\phi := \widetilde{\sigma}_3 \circ \sigma_3^{-1} : \mathbb{D}(0,r_3) \to \mathbb{D}(0,\widetilde{r}_3)$. We show that ϕ conjugates ξ_3 to $\widetilde{\xi}_3$. Indeed,

$$\begin{aligned}
\widetilde{\xi}_3 \circ \phi(\sigma_3 \circ \pi_3) &= \widetilde{\xi}_3 \circ \widetilde{\sigma}_3 \circ \pi_3 \\
&= \widetilde{\sigma}_3 \circ \pi_3 \circ T_1 \\
&= \phi \circ \sigma_3 \circ \pi_3 \circ T_1 \\
&= \phi \circ \xi_3(\sigma_3 \circ \pi_3).
\end{aligned}$$

Writing $\sigma_3 \circ \pi_3(z) = w$, we see that ϕ is a conjugacy. Recall that ξ_3 is the restriction of ξ_1, and $\widetilde{\xi}_3$ is the restriction of ξ_2. So we have proved that ξ_1 and ξ_2 are conformally conjugate in a neighborhood of 0, and the conjugating map is $\phi = \sigma_2|_{\mathbb{D}(0,\widetilde{r}_3)} \circ \left(\sigma_1|_{\mathbb{D}(0,r_3)}\right)^{-1}$.

□

With this result, we conclude that the multiplier at 0 of the by T_1 induced map does not depend on the choice of the construction. In the next subsection, we will find that multiplier.

5.3 Finding the multiplier

Theorem 5.4. *For any given construction $(z_0, \mathcal{R}, \pi, \mathcal{C}, \sigma, \xi)$, $\xi(z) = \mu z + o(z^2)$ near 0, where $\mu = e^{-\frac{2\pi i}{\beta}}$.*

Proof. Recall the choice $\operatorname{Re}\beta < 0$. For simplicity, we assume $\operatorname{Re}\beta < -1$. Set $\mu := \xi'(0)$. Suppose $\operatorname{Im}\beta \neq 0$. Then by Schwarz Lemma, $|\mu| \neq 1$, and ξ can be expressed as

$$\xi(z) = \mu z + o(z^2) \text{ around } 0.$$

Since $|\mu| \neq 1$, there exists a linearizing coordinate Φ satisfying $\xi \circ \Phi = \mu\Phi$. For any $\Lambda \in \log\mu$, by $z \mapsto e^{z\Lambda}$, we lift the dynamics $z \mapsto \mu z$ on $\mathbb{C}^*$ to $z \mapsto z+1$ on $\mathbb{C}$ with the Deck transformation

$$T_{-\frac{2\pi i}{\Lambda}}(z) := z - \frac{2\pi i}{\Lambda}.$$

Since $\Phi \circ \sigma \circ \pi$ is holomorphic, $\mathcal{R}$ is simply connected and $z \mapsto e^{\Lambda z}$ is a holomorphic covering map, there exists a holomorphic lift

$$\chi : \mathcal{R} \to \mathbb{C},$$

such that the following diagram commutes (see Theorem **??** for the existence of such lifts).

$$\begin{tikzcd}
 & \mathbb{C} \arrow[d, "e^{\Lambda z}"] \\
\mathcal{R} \arrow[ur, "\chi"] \arrow[r, "\Phi\circ\sigma\circ\pi"'] & \mathbb{C}^*
\end{tikzcd}$$

Now, we will choose Λ, so that χ satisfies the following properties:

i. $\chi \circ T_1 = T_1 \circ \chi$, and

ii. $\chi \circ H = T_{-\frac{2\pi i}{\Lambda}} \circ \chi$, where $T_{-\frac{2\pi i}{\Lambda}}(z) := z - \dfrac{2\pi i}{\Lambda}$.

Take $w_0 \in \mathcal{R}$ such that $w_1 := T_1(w_0) \in \mathcal{R}$. Take a simple curve $\gamma \in \mathcal{R}$ which joins w_0 to w_1. Under $\Phi \circ \sigma \circ \pi$, γ maps to the curve $\widetilde{\gamma} := \Phi \circ \sigma \circ \pi(\gamma)$ which joins the points $\widetilde{w}_0 := \Phi \circ \sigma \circ \pi(w_0)$ and $\widetilde{w}_1 := \Phi \circ \sigma \circ \pi(w_1)$, where $\widetilde{w}_1 = \mu \widetilde{w}_0$.

We lift $\widetilde{\gamma}$ by $z \mapsto e^z$ to the curve $\widehat{\gamma}$, and denote the endpoints by $\widehat{w}_0$ and $\widehat{w}_1$. Define

$$\Lambda := \widehat{w}_1 - \widehat{w}_0. \tag{13}$$

Here Λ is a holomorphic function of w_0. Moreover observe that

$$e^{\Lambda} = e^{\widehat{w}_1 - \widehat{w}_0} = \frac{e^{\widehat{w}_1}}{e^{\widehat{w}_0}} = \frac{\widetilde{w}_1}{\widetilde{w}_0} = \mu,$$

so that Λ is a logarithm of μ. This means it takes values in a discrete set. Therefore, Λ is constant.

The linear map $z \mapsto \Lambda z$ conjugates T_1 to $z \mapsto z + \Lambda$ in $\mathbb{C}$. It can be seen by defining $\gamma^* := \frac{1}{\Lambda}\widehat{\gamma}$ with endpoints w_0^* and w_1^*. Observe that $w_1^* = w_0^* + 1$, by construction. Hence we have seen that, with the choice of Λ given by (13), i. is satisfied.

Now we will show that ii. is satisfied with this choice of Λ. With the same

approach, take a curve Γ connecting $z_0 \in \mathcal{R}$ to $z_1 := H(z_0) \in \mathcal{R}$. Under $\Phi \circ \sigma \circ \pi$, Γ maps to a closed curve $\widetilde{\Gamma}$. By $z \mapsto e^z$, $\widetilde{\Gamma}$ lifts to $\widehat{\Gamma}$ with endpoints $\widehat{z}_0$ and $\widehat{z}_1$, satisfying $\widehat{z}_1 - \widehat{z}_0 = -2\pi i$ (since $\operatorname{Re}\beta < 0$). The linear map lifts $\widehat{\Gamma}$ to $\Gamma^* := \frac{1}{\Lambda}\widehat{\Gamma}$ with endpoints satisfying $z_1^* = z_0^* - \frac{2\pi i}{\Lambda}$. Hence, we have shown ii. is satisfied.

Observe that by i., χ extends to a 1-periodic region in an upper half plane, and χ must be of the form:

$$z \mapsto z + \alpha + o(1), \quad \text{for some } \alpha \in \mathbb{C}.$$

Then, we have

$$\chi \circ H = z + \beta + \alpha + o(1), \qquad \text{and} \tag{14}$$

$$T_{-\frac{2\pi i}{\Lambda}} \circ \chi(z) = z - \frac{2\pi i}{\log \mu} + \alpha + o(1). \tag{15}$$

By ii., (14) and (15) are equal. Therefore, we obtain $\beta = -\frac{2\pi i}{\log \mu}$.

Note that we have obtained this identity assuming $\operatorname{Im}\beta \neq 0$. We will show that this holds also for $\operatorname{Im}\beta = 0$, by analytic continuation. In order to use an analytic continuation argument, we need to show that the multiplier μ is actually a holomorphic function of β. With this information, since μ coincides with $e^{-\frac{2\pi i}{\beta}}$ except for $\operatorname{Im}\beta = 0$, we conclude that $\mu = e^{-\frac{2\pi i}{\beta}}$ everywhere. To this end, the theory of quasiconformal mappings can be invoked. We will use a similar approach as in [12, Prop 2.5.2 (ii)]. Set $S_{\frac{1}{\beta}}(z) := \frac{z}{\beta}$. The map $S_{\frac{1}{\beta}}$ conjugates H on $\mathcal{M}$ to $G(z) := z + 1 + o(1)$. In particular, $S_{\frac{1}{\beta}}$ maps fundamental domains for H to fundamental domains for G. Set $\widehat{\mathcal{R}} := S_{\frac{1}{\beta}}(\mathcal{R})$. For all $w \in \widehat{\mathcal{R}}$

$$|G(w) - w - 1| < \frac{1}{4}. \tag{16}$$

Indeed, for $S_{\frac{1}{\beta}}(z) = w$, $\frac{H(z)}{\beta} = G(w)$. Then

$$\begin{aligned} G(w) - w - 1 &= G(w) - \frac{z}{\beta} - 1 \\ &= \frac{H(z)}{\beta} - \frac{z}{\beta} - 1 \\ &= \frac{1}{\beta}(H(z) - z - \beta). \end{aligned}$$

Hence $|G(w)-w-1| = |\frac{1}{\beta}(H(z)-z-\beta)| < \frac{1}{4}$ by (11). Moreover, it is easy to see that for all $w \in \widehat{\mathcal{R}}$

$$|G'(w)-1| < \frac{1}{4},$$

using the identity $H'(z) = G'(w)$ in (12).

Since $H \circ T_1 = T_1 \circ H$, G commutes with the translation by $T_{\frac{1}{\beta}}(z) := z + \frac{1}{\beta}$, i.e., $G \circ T_{\frac{1}{\beta}} = T_{\frac{1}{\beta}} \circ G$.

We may take $\widehat{z}_0 = \widehat{x}_0 + \widehat{y}_0 := S_{\frac{1}{\beta}}(z_0)$, where z_0 is given in the construction $(z_0, \mathcal{R}, \pi, \mathcal{C}, \sigma, \xi)$ as was explained in Subsection 5.1. Set

$$\begin{aligned}
\widehat{\gamma}_0 &:= \{z;\ \operatorname{Re} z = \widehat{x}_0,\ \operatorname{Im} z \in [\widehat{y}_0, \infty)\},\\
\widehat{\gamma}_1 &:= \{z;\ \frac{\operatorname{Re}(z-\widehat{z}_0)}{\operatorname{Re}(G(\widehat{z}_0)-\widehat{z}_0)} = \frac{\operatorname{Im}(z-\widehat{z}_0)}{\operatorname{Im}(G(\widehat{z}_0)-\widehat{z}_0)},\ \operatorname{Re}\widehat{z}_0 \le \operatorname{Re} z \le \operatorname{Re} G(\widehat{z}_0)\},\\
\widehat{\gamma}_2 &:= G(\widehat{\gamma}_0).
\end{aligned}$$

Then the region in $\widehat{\mathcal{R}}$ bounded by the curve $\widehat{\gamma}_0 \cup \widehat{\gamma}_1 \cup \widehat{\gamma}_2$ is a fundamental domain, say $\mathcal{F}$ for G. Define

$$h_\beta(x+iy) = (1-x)(\widehat{z}_0+iy) + xG(\widehat{z}_0+iy) \quad \text{for } 0<x<1,\ \ y>0.$$

Observe that h_β maps $i\mathbb{R}^+$ to $\widehat{\gamma}_0$ and the half strip $\{z:\ 0<x<1,\ y>0\}$ to $\mathcal{F}$. Since G depends holomorphically on β, so does h_β. Now observe the following inequalities:

$$\begin{aligned}
\left|\frac{\partial h_\beta}{\partial \bar{z}}\right| &= \frac{1}{2}\left|\frac{\partial h_\beta}{\partial x} + i\frac{\partial h_\beta}{\partial y}\right|\\
&= \frac{1}{2}|G(\widehat{z}_0+iy) - (\widehat{z}_0+iy+1) - x(G'(\widehat{x}_0+iy)-1)|\\
&< \frac{1}{2}(\frac{1}{4}+\frac{1}{4}) = \frac{1}{4}.
\end{aligned} \tag{17}$$

$$\begin{aligned}
\left|\frac{\partial h_\beta}{\partial z} - 1\right| &= \frac{1}{2}\left|\frac{\partial h_\beta}{\partial x} - i\frac{\partial h_\beta}{\partial y} - 1\right|\\
&= \frac{1}{2}|G(\widehat{z}_0+iy) - (\widehat{z}_0+iy+1) + x(G'(\widehat{x}_0+iy)-1)|\\
&< \frac{1}{2}(\frac{1}{4}+\frac{1}{4}) = \frac{1}{4}.
\end{aligned} \tag{18}$$

By (17) and (18)

i. If $1 < |\partial h_\beta/\partial z| < \frac{5}{4}$,

$$\frac{1}{5} < |\frac{\partial h_\beta/\partial \overline{z}}{\partial h_\beta/\partial z}| < \frac{1}{4}.$$

ii. if $\frac{3}{4} < |\partial h_\beta/\partial z| < 1$,

$$\frac{1}{4} < |\frac{\partial h_\beta/\partial \overline{z}}{\partial h_\beta/\partial z}| < \frac{1}{3}.$$

In either case (i. or ii.), we have $|\frac{\partial h_\beta/\partial \overline{z}}{\partial h_\beta/\partial z}| < \frac{1}{3}$.

Let σ_0 denote the standard complex structure of $\mathbb{C}$. We obtain an almost complex structure σ_β by

- pulling back σ_0 under h_β (i.e., $\sigma_\beta = h_\beta^* \sigma_0$) on the half strip $\{z|\ \ 1 > \operatorname{Re} z > 0,\ \ \operatorname{Im} z > 0\}$, and extending 1-periodically to the upper half plane, (i.e., $\sigma_\beta = (T_1^n)^* \sigma_\beta$, $n \in \mathbb{N}$, $T_1(z) = z + 1$), and
- assigning σ_0 elsewhere (i.e., on $\{z;\ \operatorname{Im} z \leq 0\} \cup \{z;\ \operatorname{Re} z \in \mathbb{Z},\ \operatorname{Im} z \in \mathbb{R}\}$).

By construction, σ_β depends holomorphically on β. By the Integrability Theorem, there exists a unique quasiconformal mapping

$$\phi_\beta : \mathbb{C} \to \mathbb{C} \quad \text{such that} \ \ \phi_\beta^* \sigma_0 = \sigma_\beta,$$

which is normalized so that

$$\phi_\beta(0) = 0, \quad \phi_\beta(1) = 1, \quad \phi_\beta(\infty) = \infty. \tag{19}$$

Moreover ϕ_β depends holomorphically on β, since σ_β depends holomorphically on β. The map $\Phi_\beta := \phi_\beta \circ T_1 \circ \phi_\beta^{-1}$ preserves the standard complex structure (i.e., $\Phi_\beta^* \sigma_0 = \sigma_0$). So Φ_β is a conformal map of $\mathbb{C}$, that is, an affine map. Furthermore, Φ_β has no fixed points in $\mathbb{C}$ since T_1 has no fixed points in $\mathbb{C}$. Therefore Φ_β is a translation map, say $\Phi_\beta(z) = z + \alpha$, $\alpha \in \mathbb{C}$. Using (19), we have $\Phi_\beta(0) = 1$, and obtain $\alpha = 1$, that is, $\Phi_\beta = T_1$.

Now define $\psi_\beta = \phi_\beta \circ h_\beta^{-1}$ on $\mathcal{F}$. We extend ψ_β to $\widehat{\mathcal{R}}$ by

$$\psi_\beta(G(z)) = \psi_\beta(z) + 1.$$

We will show that ψ_β is conformal in $\widehat{\mathcal{R}}$ with respect to z. Observe that $z \mapsto \psi_\beta(z)$ is

- well-defined, since for each $z \in \widehat{\mathcal{R}}$, there exists a unique representative $G^n(z) \in \mathcal{F}$, $n \in \mathbb{Z}$,
- homeomorphism, since it is the composition of two quasiconformal maps, and
- analytic possibly except on $G^n(\widehat{\gamma}_1) \cap \widehat{\mathcal{R}}$ for $n \in \mathbb{Z}$.

By Morera's Theorem $z \mapsto \psi_\beta(z)$ is holomorphic near $\widehat{\gamma}_1$, hence conformal in $\widehat{R}$.

Now we will show that $\beta \mapsto \psi_\beta(z)$ is holomorphic. For $\psi_\beta \circ h_\beta(z) = \phi_\beta$, observe the following:

$$\begin{aligned}\frac{\phi_\beta(z) - \phi_{\beta_0}(z)}{\beta - \beta_0} &= \frac{\psi_\beta \circ h_\beta(z) - \psi_{\beta_0} \circ h_{\beta_0}(z)}{\beta - \beta_0} \\ &= \frac{\psi_\beta \circ h_\beta(z) - \psi_\beta \circ h_{\beta_0}(z)}{\beta - \beta_0} + \frac{\psi_\beta \circ h_{\beta_0}(z) - \psi_{\beta_0} \circ h_{\beta_0}(z)}{\beta - \beta_0}\end{aligned}$$

Taking the limit of both sides as $\beta \to \beta_0$,

$$\begin{aligned}A_1 &:= \lim_{\beta \to \beta_0} \frac{\phi_\beta(z) - \phi_{\beta_0}(z)}{\beta - \beta_0} \\ &= \lim_{\beta \to \beta_0} \frac{\psi_\beta \circ h_\beta(z) - \psi_\beta \circ h_{\beta_0}(z)}{\beta - \beta_0} + \lim_{\beta \to \beta_0} \frac{\psi_\beta \circ h_{\beta_0}(z) - \psi_{\beta_0} \circ h_{\beta_0}(z)}{\beta - \beta_0}.\end{aligned}$$

Since

$$\lim_{\beta \to \beta_0} \frac{\psi_\beta \circ h_\beta(z) - \psi_\beta \circ h_{\beta_0}(z)}{\beta - \beta_0} = \psi'_\beta(h_{\beta_0}(z)) \frac{\partial h_\beta(z)}{\partial \beta}|_{\beta = \beta_0} =: A_2,$$

$\lim_{\beta \to \beta_0} \frac{\psi_\beta \circ h_{\beta_0}(z) - \psi_{\beta_0} \circ h_{\beta_0}(z)}{\beta - \beta_0}$ exists (and equal to $A_1 - A_2$). In other words, ψ_β is holomorphic with respect to β.

Set $E(z) := e^{2\pi i z}$. The map $\widetilde{\xi}$ induced by T_1 via $E \circ \psi_\beta \circ S_{\frac{1}{\beta}}$ is univalent in a neighborhood of 0, and extends to 0 univalently, i.e., $\widetilde{\xi}(0) = 0$, $\widetilde{\xi}'(0) \neq 0$. The multiplier $\mu = \widetilde{\xi}'(0)$ is independent of the construction, by Lemma 5.2. Observe that according to this construction, $E \circ \psi_\beta \circ S_{\frac{1}{\beta}}$ is

- univalent with respect to z, and
- holomorphic with respect to β

in their domains of definition. Hence μ is a holomorphic function of the parameter β. As we have already obtained that μ coincides with $e^{-\frac{2\pi i}{\beta}}$ on the set $\{\beta; \quad \mathrm{Re}\,\beta \neq 0\}$, by the analytic continuation, we conclude that $\mu = e^{-\frac{2\pi i}{\beta}}$ everywhere. This completes the proof. $\square$

Now we give the application of Theorem 5.4, which we mentioned earlier. We will work on the form given by (1).

Proposition 5.5. *Under the hypothesis of Theorem 4.6 and after suitable normalizations of $\phi_a^\pm$, the multiplier of the fixed point 0 of each map in the holomorphic family given by (1) is equal to $e^{-\frac{2\pi i}{B}}$, where $B := B(a) = \phi_a^- - \phi_a^+$.*

Proof. Recall $\mathrm{Re}\,B < 0$. Set $T_B(z) := z + B$, and let ψ_a^- be the inverse of ϕ_a^-. The map $T_B \circ H_a^u$ is a "Deck transformation" for ψ_a^-, i.e., $\psi_a^- \circ T_B \circ H_a^u = \psi_a^-$, which can be seen as follows:

$$\begin{aligned} \psi_a^- \circ T_B \circ H_a^u &= \psi_a^- \circ T_B \circ \phi_a^+ \circ \psi_a^- \\ &= \psi_a^- \circ \phi_a^- \circ \psi_a^- \\ &= \psi_a^-. \end{aligned}$$

Possibly changing the normalization of ψ_a^- say, by a translation depending holomorphically on a, we can suppose that $H_a^u(z) = z + o(1)$, and hence $T_B \circ H_a^u(z) = z + B + o(1)$ (compare (8)).

Take a point z_0 (with sufficiently large imaginary part) in the domain of $T_B \circ H_a^u$, and obtain the region $\mathcal{R}$ as explained in Subsection 5.1. Considering the identities $\psi_a^- \circ T_1 = f_a \circ \psi_a^-$ and $\psi_a^- \circ T_B \circ H_a^u = \psi_a^-$, we see that ψ_a^- takes the role of the projection $\sigma \circ \pi$, and f_a takes the role of the univalent map ξ locally in the abstract model construction $(z_0, \mathcal{R}, \pi, \mathcal{C}, \sigma, \xi)$. So

by Theorem 5.4, we conclude that the multiplier of the fixed point 0 in the dynamics of f_a is equal to $e^{-\frac{2\pi i}{B}}$. □

Finally we prove Theorem 5.1.

of Theorem 5.1. In an open sector Λ as given in (5), the multiplier map is univalent by Proposition 3.2 , and is equal to $e^{-\frac{2\pi i}{B}}$ by Proposition 5.5. This implies, B is also univalent with respect to λ in Λ, and hence with respect to a in Δ. □

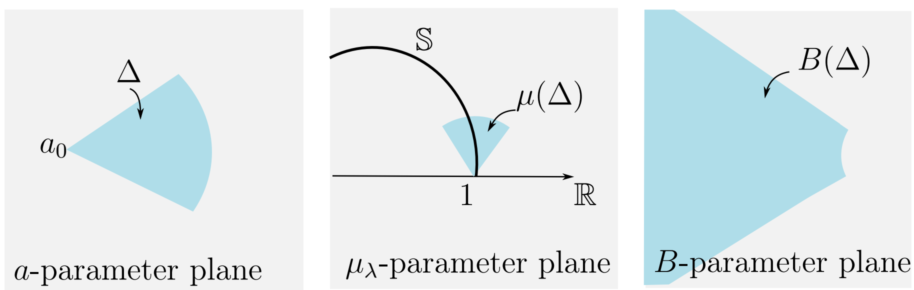

Figure 5: Correspondence of the parameter planes.

Concluding this section, we would like to point out that as B is a univalent map with respect to a in some open sector, we can assign B as the new parameter by setting $a = a(B)$ (see Figure 5). With this in mind, we are ready to prove Main Theorem.

6 Proof

The main two tools in the proof are the holomorphic motion and the (Douady)-Fatou coordinates. We use the new parameter $B = \phi_a^- - \phi_a^+$, which we have studied in the previous section. We denote by $\gamma_a[t, t+1]$, the fundamental segment with endpoints $\gamma_a(t)$ and $\gamma_a(t+1)$ on a fixed ray γ_a, and by $[\gamma_a(t), \gamma_a(t+1)]$, the line segment with endpoints $\gamma_a(t)$ and $\gamma_a(t+1)$.

Consider the holomorphic family of holomorphic maps $\{f_a\}_{a\in\mathbb{D}(a_0,r)}$ as was stated in Section 2. Suppose the forward invariant curve $\gamma_{a_0} : (-\infty, \mathcal{T}) \to \mathbb{C}$, $\mathcal{T} \in (-\infty, \infty]$ lands at the parabolic fixed point z_0. Obviously γ_{a_0} lands

through the repelling petal Ω^-. The Fatou coordinate $\phi^-_{a_0}$ maps $\gamma_{a_0} \cap \Omega^-$ to a 1-periodic ray in the Fatou coordinate plane. We extend this curve 1-periodically to $-\infty$ and $+\infty$, and denote the extension by $\widetilde{\gamma}_{a_0}$.

Let Δ be an open central sector with corner point at a_0, on which the Douady-Fatou coordinates are well defined and moreover for all $a \in \Delta$, $a \mapsto B(a)$ is univalent. Existence of such sector is guaranteed by Theorem 5.1. By central sector, we mean that the corresponding sector $\mu(\Delta)$ contains an arc of the unit circle $\mathbb{S}$ (see Figure 5). Notice that $\mu(\Delta) \cap \mathbb{S}$ is the set of points which correspond to real B values. Recall that $\phi^-_a \rightrightarrows \phi^-_{a_0}$ on compact subsets of the repelling petal Ω^- (and $\phi^+_a \rightrightarrows \phi^+_{a_0}$ on compact subsets of the parabolic basin).

Let $H : \mathbb{D}(a_0, \delta) \times \gamma_{a_0}[T, \mathcal{T}) \to \widehat{\mathbb{C}}$ be the holomorphic motion as was given in the statement of Main Theorem. Given $T' < T$, we may assume that H extends equivariantly to $\mathbb{D}(a_0, \delta_1) \times \gamma_{a_0}[T', \mathcal{T})$, for some $\delta_1 \leq \delta$, by using the dynamics. Suppose $t' \in \mathbb{R}$, such that $t' + 1 < \mathcal{T}$, $\gamma_{a_0}[t', t' + 2] \subset \Omega^-$, and $H : \mathbb{D}(a_0, \delta_1) \times \gamma_{a_0}[t', \mathcal{T}) \to \mathbb{C}$ is an equivariant extension. Possibly reducing δ_1 to δ_2, we can assume that

$$K := \bigcup_{a \in \overline{\mathbb{D}(a_0,\delta_2)}} H(a, \gamma_{a_0}[t', t' + 2]) = H(\overline{\mathbb{D}(a_0, \delta_2)}, \gamma_{a_0}[t', t' + 2]) \subset \Omega^-,$$

by continuity of the holomorphic motion.

By a further reduction of δ_2 to $0 < \delta_3 \leq \delta_2$, we can assume that K is contained in the domain of the outgoing Douady-Fatou coordinate ϕ^-_a for $a \in \Delta_3 := \Delta \cap \overline{\mathbb{D}(a_0, \delta_3)}$. For $a \in \Delta_3$, ϕ^-_a sends $\gamma_a[t', t'+2]$ to $\phi^-_a(\gamma_a[t', t'+2])$, which extends 1-periodically to $-\infty$ (and also to $+\infty$). Denote this extension by $\widetilde{\gamma}_a$ (see Figure 6).

Lemma 6.1. *There exists an isotopy of curves*

$$\begin{array}{rcl} \mathcal{I} : \overline{\Delta_3} \times [t', t' + 1] \times [0, 1] \times [0, 1] & \to & \mathbb{C} \\ (a, t'', s, t) & \mapsto & \mathcal{I}(a, t'', s, t) \end{array}$$

such that

$$\begin{array}{rcl} \mathcal{I}(a, t'', 0, [0, 1]) & = & \widetilde{\gamma}_a[t'', t'' + 1], \\ \mathcal{I}(a, t'', 1, [0, 1]) & = & [\widetilde{\gamma}_a(t''), \widetilde{\gamma}_a(t'' + 1)], \end{array}$$

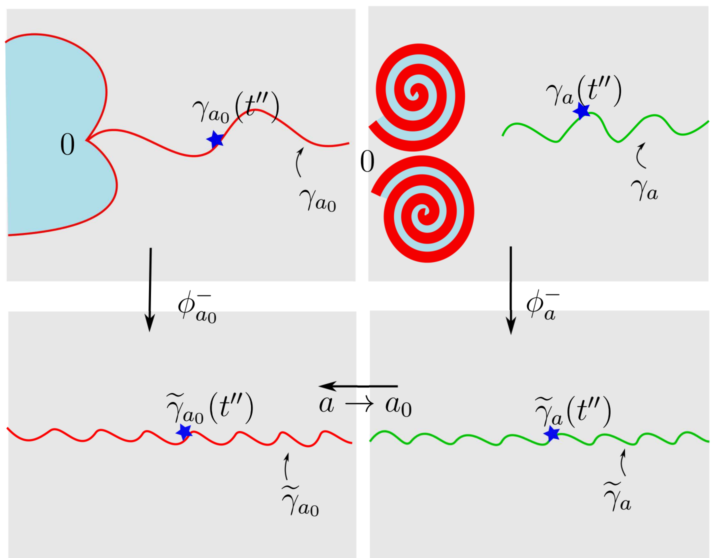

Figure 6: Projection of the rays in the (Douady)-Fatou Coordinates.

relative to endpoints $\widetilde{\gamma}_a(t'')$ and $\widetilde{\gamma}_a(t''+1)$. Moreover, this isotopy depends continuously on the parameter a and the potential t''.

Proof. Take a horizontal line $\mathcal{L} := \mathbb{R} + iy$ such that $\mathcal{L} \cap \widetilde{\gamma}_a = \emptyset$ (take y large enough so that $\mathcal{L}$ does not depend on the parameter). Denote by Ω, the domain bounded by $\mathcal{L}$ from above and by $\widetilde{\gamma}_a$ from below. Since Ω is simply connected, by the Uniformization Theorem, there exists a conformal map $\Phi_a : S \to \Omega$ from the horizontal strip $S := \{z;\quad 0 < \operatorname{Im} z < 1\}$ to Ω. Moreover, since $\partial\Omega$ is a Jordan curve, Φ_a extends continuously to $\partial\Omega$. We define an isotopy of curves as the following:

$$\mathcal{I}(a, t'', s, t) := \Phi_a\Big(is + \Phi_a^{-1}(\widetilde{\gamma}_a(t''+t))\Big) + \widetilde{\gamma}_a(t'') - \Phi_a\Big(is + \Phi_a^{-1}(\widetilde{\gamma}_a(t''))\Big).$$

In particular, $\mathcal{I}(a, t'', 0, t) = \widetilde{\gamma}_a(t''+t)$ and $\mathcal{I}(a, t'', 0, 0) = \widetilde{\gamma}_a(t'')$. It is clear that $\mathcal{I}$ is continuous with respect to t'', since $\widetilde{\gamma}_a$ is a continuous curve. Moreover $\mathcal{I}$ is continuous with respect to a in $\overline{\Delta_3}$. It is because the uniformizing parameter Φ_a and $\widetilde{\gamma}_a$ are continuous with respect to the parameter. The continuity of $\widetilde{\gamma}_a$ comes from the fact that $\gamma_a[t', t'+2] \subset K$ is

contained in the domain of the outgoing Douady-Fatou coordinate ϕ_a^-. So $\widetilde{\gamma}_a[t', t'+2] = \phi_a^-(\gamma_a[t', t'+2])$ is well-defined and depends continuously on the parameter $a \in \overline{\Delta_3}$. $\square$

Remark 6.2. Since $\overline{\Delta_3}$, $[t', t'+1]$, $[0,1]$ are compact, and $\mathcal{I}$ is continuous in (a, t'', s, t), then $\widetilde{C} = \mathcal{I}(\overline{\Delta_3} \times [t', t'+1] \times [0,1] \times [0,1])$ is also compact. If $\widetilde{C}$ is not contained in $\phi_{a_0}^-(\Omega^-)$, possibly reducing δ_3 to δ_4, and using the T_1-equivariance of $\mathcal{I}$, i.e.,

$$\mathcal{I}(a, t''-1, s, t) = \mathcal{I}(a, t'', s, t) - 1,$$

we can assume $\widetilde{C} - n \subset \phi_{a_0}^-(\Omega^-)$. Finally, if $\widetilde{C} - n$ is not in the domain of $\psi_a^- = (\phi_a^-)^{-1}$ for $a \in \Delta_4 := \Delta \cap \mathbb{D}(a_0, \delta_4)$, with a further reduction δ_5 from δ_4, we can assume that $\widetilde{C} - n$ is in the domain of ψ_a^- for $a \in \Delta_5 = \Delta \cap \mathbb{D}(a_0, \delta_5)$, and $H : \mathbb{D}(a_0, \delta_5) \times \gamma_{a_0}[t'-n, \mathcal{T}]$ is an equivariant extension. For $a \in \Delta_5$ and $t'' \in [t'-n, t'-n+1]$, denote by $\widetilde{C}_{a,t''}$, the restriction $\mathcal{I}|_{\{a\}\times\{t''\}\times[0,1]\times[0,1]}$.

of Main Theorem - Conclusion. For $a \in \overline{\Delta_5}$, let $\widetilde{\gamma}_{a_0}$ and $\widetilde{\gamma}_a$ denote the extensions of $\phi_{a_0}^-(\gamma_{a_0}[t', \mathcal{T}]) \cap \Omega^-$ and $\phi_a^-(\gamma_a[t', t'+2])$, respectively. Recall that $B = \phi_a^- - \phi_a^+$ is constant for fixed a. Moreover, for the choice of normalizations of $\phi_a^\pm$ so that $H_a^u(z) = z + o(1)$, $B = \phi_a^- - \phi_a^+$ is univalent with respect to the parameter $a \in \Delta$ (see Section 5). So we can write $a = a(B)$. In order to fix normalization completely, in addition, we shall suppose $B := \phi_a^-(s(a))$ (then $\phi_a^+(s(a)) = 0$, by the choice of normalization).

By the continuity of $\widetilde{\gamma}_a(t'')$ on the parameter at a_0 (see Lemma 6.1), we can define

$$\epsilon := \sup_{t'' \in [t'-n, t'-n+1], a \in \overline{\Delta_5}} |\widetilde{\gamma}_a(t'') - \widetilde{\gamma}_{a_0}(t'')|. \tag{20}$$

Choose t_0 with the property that, for all $t \leq t_0$:

$$\widetilde{\gamma}_{a_0}(t) \in B(\Delta_5) \quad \text{ and } \quad d_E(\widetilde{\gamma}_{a_0}(t), \partial B(\Delta_5)) > \epsilon.$$

We want to compare $\widetilde{\gamma}_a(t)$ and B. Consider the following identity:

$$\widetilde{\gamma}_a(t) - B = \Big(\widetilde{\gamma}_a(t) - \widetilde{\gamma}_{a_0}(t)\Big) + \Big(\widetilde{\gamma}_{a_0}(t) - B\Big). \tag{21}$$

First we will show that whenever $a \in \Delta_5$, then

$$|\widetilde{\gamma}_a(t) - \widetilde{\gamma}_{a_0}(t)| < \epsilon. \tag{22}$$

Let k be the natural number such that $t + k := t'' \in [t' - n, t' - n + 1]$. By 1-periodicity,

$$\widetilde{\gamma}_a(t) + k = \widetilde{\gamma}_a(t + k) = \widetilde{\gamma}_a(t''),$$

and so

$$\widetilde{\gamma}_a(t) - \widetilde{\gamma}_{a_0}(t) = \widetilde{\gamma}_a(t'') - \widetilde{\gamma}_{a_0}(t'').$$

Therefore (20) implies (22).

Set $B_0 := \widetilde{\gamma}_{a_0}(t)$, and define holomorphic functions of B in $B(\Delta_5)$,

$$\begin{aligned} \xi_1(B) &:= \widetilde{\gamma}_{a(B)}(t) - \phi^-_{a(B)}\Big(s(a(B))\Big) \quad \text{and} \\ \xi_2(B) &:= B_0 - B. \end{aligned}$$

The map ξ_1 is a holomorphic function of B because the Douady-Fatou coordinates and the singular value depends holomorphically on the parameter a in Δ_5, and hence on the parameter B in $B(\Delta_5)$. Obviously ξ_2 is a holomorphic map of B. In terms of ξ_1 and ξ_2, (21) can be written as

$$\xi_1(B) = \Big(\widetilde{\gamma}_a(t) - \widetilde{\gamma}_{a_0}(t)\Big) + \xi_2(B).$$

This yields by (22)

$$|\xi_1(B) - \xi_2(B)| < \epsilon \quad \text{for } \; B \in B(\Delta_5).$$

For $B \in \mathbb{S}(B_0, \epsilon) := \partial\mathbb{D}(B_0, \epsilon)$, $|\xi_2(B)| = \epsilon$. Since ξ_2 has one simple zero at B_0, ξ_1 has a single simple zero in $\mathbb{D}(B_0, \epsilon)$ by Rouché's Theorem (see Figure 7). This means, in $\mathbb{D}(B_0, \epsilon)$, there exists a unique parameter B', which satisfies

$$\phi^-_{a(B')}\Big(s(a(B'))\Big) = \widetilde{\gamma}_{a(B')}(t).$$

Set $a' := a(B')$. We claim that $\gamma_{a'}$ has a unique $f_{a'}$-invariant extension until it hits the singular value $s(a')$, and that

$$s(a') = \gamma_{a'}(t).$$

We use the fact that the fundamental segment $\widetilde{\gamma}_{a'}[t'', t'' + 1]$ and the line segment $[\widetilde{\gamma}_{a'}(t''), \widetilde{\gamma}_{a'}(t'' + 1)]$ are isotopic relative to endpoints $\widetilde{\gamma}_{a'}(t'')$ and

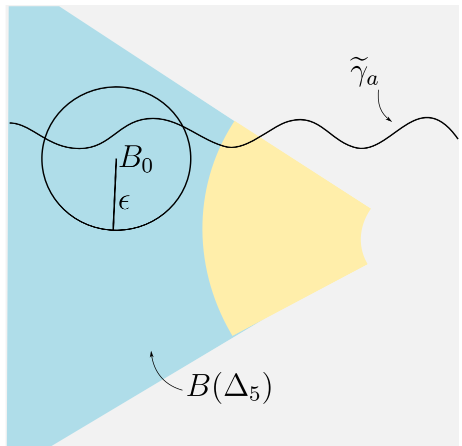

Figure 7: B-plane.

$\widetilde{\gamma}_{a'}(t''+1)$. This result is given by Lemma 6.1. Moreover, the line segment $\widetilde{l} = [\phi^-_{a'}(s(a')), \widetilde{\gamma}_{a'}(t'')]$ is contained in the domain of the Douady-Fatou parameter $\psi^-_{a'} = (\phi^-_{a'})^{-1}$. So $l := \psi^-_{a'}(\widetilde{l})$ is well defined and is a simple $f_{a'}$-invariant curve which connects $s(a')$ and $\gamma_{a'}(t'')$. On the other hand, the isotopy of curves $\widetilde{C}_{a',t''}$ maps to an isotopy of curves by $\psi^-_{a'}$ in the dynamical plane for $f_{a'}$. By pulling back $\psi^-_{a'}([\widetilde{\gamma}_{a'}(t''), \widetilde{\gamma}_{a'}(t''+1)])$ n times we reach $s(a')$.

Note that $\psi^-_{a'}([\widetilde{\gamma}_{a'}(t''), \widetilde{\gamma}_{a'}(t''+1)])$ is in the same isotopy class as $\psi^-_{a'}(\widetilde{\gamma}_{a'}[t'', t''+1]) = \gamma_{a'}[t'', t''+1]$ relative to the set $\bigcup_{i=0}^{n+1} f_{a'}(s(a'))$. So, by pulling back $\gamma_{a'}[t'', t''+1]$ under $f_{a'}$ n times, we reach $s(a')$ as well. This gives us a well-defined extension of $\gamma_{a'}$ until it hits $s(a')$. In other words, the singular value $s(a')$ is on the invariant curve $\gamma_{a'}$ at potential t.

This relation induces a map Γ which assigns to each potential, a unique parameter. More precisely, $\Gamma : (-\infty, t_0] \to \overline{\Delta_5}$, so that writing $a' = \Gamma(t)$ then $s(a') = \gamma_{a'}(t)$.

Now we will show that Γ forms a curve. The map

$$B(t) := \oint_{\mathbb{S}(B_0,\epsilon)} B \frac{\xi_1'(B)}{\xi_1(B)} dB$$

is continuous with respect to t. Since $a(B(t)) = \Gamma(t)$, then Γ is a continuous curve, as it is parametrized by the continuous curve $(-\infty, t_0]$.

Finally we show that $\Gamma(t)$ lands at a_0. As $t \to -\infty$, $\gamma_{a_0}(t) \to z_0$, and $\widetilde{\gamma}_a \to -\infty$. This implies $\gamma_a(t) = B = \phi_a^-(s(a)) \to -\infty$, and equivalently $a \to a_0$. Hence we obtain as $t \to -\infty$, $\Gamma(t) \to a_0$. Therefore we conclude that the curve Γ lands at the parabolic parameter a_0. This completes the proof.

□

7 Application: On parameter rays for families of transcendental entire maps

We denote by $\widehat{\mathcal{B}}$ the class of transcendental entire maps with bounded singular set, of finite order, or finite composition of such maps. Existence of dynamic ray structure is proved in [10] for this class of maps. The following converse landing theorem is useful to fulfill the hypothesis of the transcendental entire version of Main Theorem.

Theorem 7.1. *(Benini-Fagella, [1, Cor 7.1]) Let $f \in \widehat{\mathcal{B}}$, and assume all periodic rays land. If z_0 is a parabolic point, then in each repelling petal there exists at least one periodic dynamic ray landing at z_0.*

Define the set of singular values of $f \subset \widehat{\mathcal{B}}$, denoted by $\mathcal{S} := \mathcal{S}(f)$, as the union of its critical values, its asymptotic values (i.e. values $a \in \mathbb{C}$ for which there exists a curve tending to infinity whose image lands at a), and their accumulation points. Define the post-singular set by

$$\mathcal{P} = \overline{\bigcup_{n \geq 0} f^n(\mathcal{S})}.$$

The following theorem is the main result of [4], which states a landing theorem in dynamical spaces.

Theorem 7.2. *For $f \in \widehat{\mathcal{B}}$ with bounded post-singular set $\mathcal{P}$, all periodic dynamic rays land, and the landing points are either repelling or parabolic periodic points.*

So, for maps in the class $\widehat{\mathcal{B}}$ with bounded post-singular set, all parabolic points are landing points of periodic dynamic rays by Theorem 7.1 together

with Theorem 7.2. We will restrict our work to this type of maps.

Now consider a one parameter holomorphic family $\{f_a\}_{a\in M}$ of entire maps of class $\widehat{\mathcal{B}}$. We denote a dynamic ray given by some external address $\underline{s}$ in the dynamical plane for f_a by $g^a_{\underline{s}}$.

Suppose

- For $a_0 \in M$, f_{a_0} has bounded post-singular set,
- f_{a_0} has a nondegenerate and nonpersistent parabolic fixed point at z_0 with multiplier 1,
- the fixed ray $g^{a_0}_{\underline{s}}$ landing at the parabolic fixed point z_0 (see Theorem 7.2) does not contain a critical point.
- there exists $r, R > 0$, such that for all $a \in \mathbb{D}(a_0, r)$, f_a has exactly one singular value $s(a)$ in $\overline{\mathbb{D}(z_0, R)}$ which depends holomorphically on a, and
- the attracting petal of the parabolic fixed point z_0 contains only $s(a_0)$ of all singular values (if there are more).

Theorem 7.3. *(A converse landing theorem for fixed parameter rays) In the setting above, there exists a simple bounded curve $G_{\underline{s}}(-\infty, t_0] \to \mathbb{C}$, $t_0 < \infty$ in the parameter plane, which consists of parameters with the property that for $a = G_{\underline{s}}(t)$, there exists a fixed ray $g^a_{\underline{s}}$ with $s(a) = g^a_{\underline{s}}(t)$. Moreover, $G_{\underline{s}}$ lands at a_0.*

We remind again that we state the theorem for fixed dynamics rays, which we can also apply for the qth iterate of a q-periodic ray, as mentioned in Introduction section.

Proof. By Proposition 1.1, there exists a neighborhood Ω of a_0 in the parameter plane, and a holomorphic motion H of $g^{a_0}_{\underline{s}}[T, \infty)$ for large potentials $T > 0$, which is parametrized over Ω. Hence all the assumptions of Main Theorem are fulfilled. Thus, for the parabolic parameter a_0 and the dynamic ray $g^{a_0}_{\underline{s}}$, there exists a curve G, which consists of the parameter values a, for which in the dynamical plane for f_a, the singular value $s(a)$ is on the fixed ray $g^a_{\underline{s}} := H(a, g^{a_0}_{\underline{s}})$. For such curve G, we can use the notation $G_{\underline{s}}$ in order to indicate the relation with $g^a_{\underline{s}}$. Moreover $\lim_{t\to-\infty} G_{\underline{s}}(t) = a_0$ is also guaranteed by Main Theorem. $\square$

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